# Assignment 3

This homework is due *Tuesday* Sep 27.

There are total 54 points in this assignment. 42 points is considered 100%. If you go over 42 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.2–2.4 in Bartle–Sherbert.

## 1. Quick cheat-sheet.

Reminder. (Subsection 2.1.5 in textbook) Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D (for example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). The set $\mathbb{P} \subset \mathbb{A}$ is called the set of *positive elements* if

(i) If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$,
(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$,
(iii) If $a \in \mathbb{A}$, then exactly one of the following holds: $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$.

Then $a < b$ if and only if $b - a \in \mathbb{P}$; $a \leq b$ if and only if $b - a \in \mathbb{P} \cup 0$.

One can prove the following (Theorem 2.1.7 in textbook):

Theorem. Let $a, b, c \in \mathbb{A}$.

(a) If $a > b$ and $b > c$ then $a > c$,
(b) if $a > b$, then $a + c > b + c$,
(c) if $a > b$, $c > 0$, then $ca > cb$,
if $a > b$, $c < 0$, then $ca < cb$.

## 2. Exercises.

(1) (2.1.10, 11b) For $a, b, c, d \in \mathbb{R}$, prove that
   (a) [2pt] if $a < b,\ c \leq d$, then $a + c < b + d$,
   (b) [2pt] if $0 < a < b,\ 0 < c \leq d$, then $0 < ac < bd$,
   (c) [2pt] if $a < b$, then $a < \frac{1}{2}(a + b) < b$.

   Every inequality you write should be accompanied by a reference to the exact property (i)–(iii), Theorem above or a previously proved claim that you are using.

(2) [2pt] (2.1.12) Let $a, b, c, d \in \mathbb{R}$ satisfy $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, an example where $ac > bd$, and an example where $ac = bd$.

(3) [3pt] (2.1.19) Prove that $\left(\frac{1}{2}(a + b)\right)^2 \leq \frac{1}{2}\left(a^2 + b^2\right)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

— *see next page* —

(4) In each case below, determine if $P$ is a set of positive elements (i.e. whether it satisfies (i), (ii) and (iii)).
  (a) [2pt] $\mathbb{A} = \mathbb{Z}$, $P = \mathbb{N}$,
  (b) [2pt] $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,
  (c) [2pt] $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,
  (d) [3pt] $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,
  (e) [4pt] Prove that for $\mathbb{A} = \mathbb{C}$, there is no set of positive elements. (In other words, one cannot imbue $\mathbb{C}$ with a meaningful order.)

Note: items 4d, 4e deal with complex numbers that not everyone may be familiar with. So, these two questions are excluded from denominator of the grade for this homework. They are nevertheless included in numerator and you are encouraged to attempt them.

(5) (Ex. 2.2.12bd,13bd) Determine and sketch the set of pairs $(x, y)$ in $\mathbb{R} \times \mathbb{R}$ that satisfy
  (a) [2pt] $|x| + |y| = 2$.
  (b) [2pt] $|x| - |y| = 2$.
  (c) [2pt] $|x| + |y| \geq 2$.
  (d) [2pt] $|x| - |y| \geq 2$.

(6) [3pt] (Ex. 2.2.11) Find all $x \in \mathbb{R}$ that satisfy both $|2x-3| < 5$ and $|x+1| > 2$ simultaneously.

(7) (a) [2pt] Let $S \subset \mathbb{R}$ be a bounded set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \leq \sup S$.
  (b) [2pt] (Ex. 2.3.9) Show that if $A$ and $B$ are bounded nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup A \cup B = \sup\{\sup A, \sup B\}$.
  (c) [3pt] (Ex. 2.4.6) For $A, B$ as in previous item, show that $A + B = \{a + b : \ a \in A, b \in B\}$ is a bounded set. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.
  (d) [3pt] Find $\sup\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\sup\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$. (*Hint:* for the last two questions, use the previous item 7c.)
  (e) [3pt] For $A, B$ as in item 7c, show that $AB = \{ab : \ a \in A, b \in B\}$ is a bounded set. Is it true that always $\sup AB = \sup A \cdot \sup B$?

(8) (2.4.4) Let $S$ be a nonempty and bounded subset of $\mathbb{R}$.
  (a) [3pt] Let $a > 0$, and let $aS = \{as | s \in S\}$. Prove that
$$\inf(aS) = a \inf S, \qquad \sup(aS) = a \sup S.$$
  (b) [3pt] Let $b < 0$, and let $bS = \{bs | s \in S\}$. Prove that
$$\inf(bS) = b \sup S, \qquad \sup(bS) = b \inf S.$$